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OPTIMALITY OF SPLINE ALGORITHMS

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ABSTRACT

We study algorithms were studied for approximating Sf where S is a linear operator and f is an element of a set. ~~We introduce~~ the concept of spline algorithms is introduced and ~~establish~~ are established optimality properties of these algorithms. This unifies and generalizes many known results.

1. INTRODUCTION

We wish to approximate Sf where S is a linear operator and f belongs to a set $\mathfrak{F}_0$. Assume we are given $\mathfrak{N}f$ where $\mathfrak{N}$ is a finite dimensional linear operator. We say $\mathfrak{N}$ is an information operator and that $\mathfrak{N}f$ is the information on f. We approximate Sf by $\varphi(\mathfrak{N}f)$ where φ is an operator which is called an algorithm.

There are many papers dealing with optimal algorithms, i.e., algorithms φ for which the global error $e(\varphi) = \sup\{\| Sf-\varphi(\mathfrak{N}f)\| : f \in \mathfrak{F}_0\}$ is minimized. The optimal algorithms produce the best possible approximations for "worst" elements f. By a worst f we mean that $\| Sf-\varphi(\mathfrak{N}f)\| = e(\varphi)$. However it may happen that an optimal algorithm φ does not produce a best possible approximation for "easy" elements f. (By an easy f we mean $\| Sf-\varphi(\mathfrak{N}f)\| \ll e(\varphi)$.) For the user who may want to solve the problem for just such an easy f this is a very undesirable property. Therefore in this paper we study algorithms which are not only optimal (or nearly optimal) but for which the local error $\| Sf-\varphi(\mathfrak{N}f)\|$ is almost as small as possible for every f from $\mathfrak{F}_0$.

The algorithms which have the smallest possible local error for every f are called central and were introduced by Traub and Woźniakowski [77]. We introduce the concept of deviation $\mathrm{dev}(\varphi)$ of an algorithm φ. This is the ratio, for the worst case f, between the local error of φ and the error of a central algorithm. Thus, $\mathrm{dev}(\varphi) \in [1,+\infty]$. We are interested in algorithms with small deviation.

In general, optimal algorithms have large combinatory complexity, i.e., the computation of $\varphi(y)$ given $y = \mathfrak{N}f$ requires much more work than the computation of $\mathfrak{N}f$. There exists a class of algorithms with small combinatory

complexity. This is the class of linear algorithms φ (i.e., φ is a linear operator). Since we want to assure small combinatory complexity we shall study linear algorithms in this paper.

These twin desiderata of small deviation and linearity suggest the following questions:

(1.1) Do there exist linear algorithms with small deviation?

(1.2) Do there exist linear algorithms with small deviation which are optimal?

We introduce the concept of a spline algorithm and show that spline algorithms permit us to answer (1.1) and (1.2).

Splines are extensively used in numerical mathematics and in the theory of approximation. There are enormous numbers of papers dealing with many theoretical and practical aspects of splines. Many optimal properties of splines are known. Probably Schoenberg [64a] was the first one who realized the close connection between splines and optimal algorithms in the sense of Sard.

Splines were used to establish optimal algorithms (sometimes in the sense of Sard) for many problems. For instance, see Coman [72], Karlin [71], Kornejčuk and Lušpaj [63], Kornejčuk [74], Lee [77], Ligun [76], Lipow [73], Schoenberg [64b, 65, 69, 70] and Secrest [65b] who considered the integration problem, Bojanov [75], Forst [77], Gaffney and Powell [76], Gaffney [77a, 77b] who considered the interpolation problem, de Boor [77], Golomb [77], Melkman [77], Micchelli, Rivlin and Winograd [76], Micchelli and Pinkus [77] who considered the approximation problem, Ahlberg and Nilson [66], Mangasarian

and Schumaker [73], Mansfield [72], Nielson [73], Reinsch [74], Ritter [70], Schoenberg [64a] and Secrest [65a] who considered approximation of linear functionals, Grebennikov and Morozov [77] and Micchelli and Rivlin [77] who considered approximation of linear operators, and Grebennikov [78] who considered approximation of nonlinear operators. Also a classic paper of Golomb and Weinberger [59] dealing with approximation of linear functionals implicitly made use of optimal properties of splines.

This paper unifies and generalizes many known results and develops new optimality properties of spline algorithms.

We summarize the results of the paper. In Section 2 we briefly state the problem, the notation, and fundamental known results. We introduce the concept of deviation which plays a fundamental role in the paper. In Section 3 we recall the definition of splines in a linear normed space. We give a general definition of a spline algorithm in Section 4. Spline algorithms are homogenous and not necessarily linear. We prove their deviation is no greater than two. Assuming the uniqueness of the spline algorithm we prove that any linear (in fact, even any homogenous) non-spline algorithm has infinite deviation. This yields the answer to (1.1). Namely, the class of linear algorithms with finite deviation consists only of linear spline algorithms. So, this class is empty iff spline algorithms are nonlinear. In Section 5 we specialize to spline algorithms in a Hilbert case. We show that then the spline algorithm is linear and central. This means that its deviation is equal to unity and its combinatory complexity is proportional to the "amount" of information. Section 6 returns to the general case. We give necessary and sufficient conditions for a spline algorithm to be central or

optimal. This answers (1.2). We illustrate the analysis by several examples which shows the sharpness of our lemmas and theorems. In Section 7 we briefly discuss a relation between the Kolmogorov n-width and the radius of information. Section 8 contains a summary of the paper.

This paper is closely related to Traub and Woźniakowski [77]. Although we define all concepts required here, it would probably be helpful to the reader to be familiar with the concepts introduced in that paper.

2. BASIC CONCEPTS AND DEFINITIONS

We recall and slightly extend the basic notation and some results of Traub and Woźniakowski [77].

Let S and T be linear operators such that

$$(2.1)\qquad \begin{aligned} S&: \mathfrak{I}_1 \to \mathfrak{I}_2, \\ T&: \mathfrak{I}_1 \to \mathfrak{I}_4 \end{aligned}$$

where $\mathfrak{I}_1$ is a linear space and $\mathfrak{I}_2, \mathfrak{I}_4 = T(\mathfrak{I}_1)$ are linear and normed spaces over the real or complex field. Our aim is to approximate an element $\alpha = S(f)$ for any $f \in \mathfrak{I}_0$ where

$$(2.2)\qquad \mathfrak{I}_0 = \{f \in \mathfrak{I}_1 : \|Tf\| \le 1\}.$$

To approximate α we assume we know the <u>information</u> operator $\mathfrak{N}(f) = [L_1(f), L_2(f), \ldots, L_n(f)]$ where $L_1, L_2, \ldots, L_n$ are linearly independent linear functionals. The number n is called the cardinality of the information operator $\mathfrak{N}$ and is denoted by $n = \text{card}(\mathfrak{N})$.

By an <u>algorithm</u> φ we mean an operator which approximates $\alpha = S(f)$ knowing only $y = \mathfrak{N}(f) \in \mathbb{C}^n$, i.e., $\varphi: D_\varphi \subset \mathbb{C}^n \to \mathfrak{I}_2$. Note that $\varphi(y)$ has to approximate all elements $S(f)$ for any $f \in \mathfrak{I}_0$ such that $\mathfrak{N}(f) = y$. Define two sets

$$(2.3)\qquad V(y) = \{f \in \mathfrak{I}_0 : \mathfrak{N}(f) = y\},$$

$$(2.4)\qquad U(y) = \{Sf : f \in V(y)\}.$$

Then an algorithm φ cannot recognize which element $Sf \in U(y)$ is actually approximated. The <u>local error</u> $e(\varphi, y)$ of the algorithm φ for $y \in \mathfrak{N}(\mathfrak{I}_0)$ is

$$(2.5)\quad e(\varphi,y) = \sup_{f\in V(y)} \| \varphi(y)-Sf\| .$$

By the global error we mean

$$(2.6)\quad e(\varphi) = \sup_{y\in\mathfrak{N}(\mathfrak{F}_0)} e(\varphi,y) = \sup_{f\in\mathfrak{F}_0} \| \varphi(\mathfrak{N}(f))-Sf\| .$$

Let $\Phi = \Phi(\mathfrak{N},S,T)$ be the class of all algorithms which use the information operator $\mathfrak{N}$ for the problem defined by S and T. Traub and Woźniakowski [77] proved

Theorem 2.1

$$(2.7)\quad \inf_{\varphi\in\Phi} e(\varphi,y) = \operatorname{rad} U(y), \qquad \forall y \in \mathfrak{N}(\mathfrak{F}_0)$$

$$(2.8)\quad \inf_{\varphi\in\Phi} e(\varphi) = r(\mathfrak{N},S,T)$$

where rad U(y) denotes the radius of the set U(y),

$$(2.9)\quad \operatorname{rad} U(y) = \inf_{x\in\mathfrak{F}_2} \sup_{f\in V(y)} \| x-Sf\| ,$$

and

$$(2.10)\quad r(\mathfrak{N},S,T) = \sup_{y\in\mathfrak{N}(\mathfrak{F}_0)} \operatorname{rad} U(y) = c \sup_{h\in\ker\mathfrak{N}} \frac{\|Sh\|}{\|Th\|}$$

is called the radius of information $\mathfrak{N}$ for the problem S,T, and c is a constant from [1.2]. ∎

See also Micchelli and Rivlin [77] where some related results may be found. We shall assume that $r(\mathfrak{N},S,T) < +\infty$. As in Traub and Woźniakowski [77] observe that if $\ker \mathfrak{N} \cap \ker T \not\subset \ker S$ then there exists an element $h \in \ker \mathfrak{N} \cap \ker T$, $Sh \neq 0$ and $Sf + cSh \in U(y)$ for any y and any constant c

where $\mathfrak{N}(f) = y$ and $f \in \mathfrak{J}_0$. Hence $\ker \mathfrak{N} \cap \ker T \not\subset \ker S$ implies $\text{rad}\, U(y) = +\infty$, $\forall y \in \mathfrak{N}(\mathfrak{J}_0)$. Therefore we shall assume throughout the paper that

$$(2.11) \quad \ker \mathfrak{N} \cap \ker T \subset \ker S$$

which is a necessary condition for $r(\mathfrak{N},S,T) < +\infty$.

From (2.8) it follows that an algorithm φ is optimal iff $e(\varphi) = r(\mathfrak{N},S,T)$. As we mentioned in the Introduction we are primarily interested in algorithms which produce a nearly optimal approximation for every f. Theorem 2.1 states that the radius rad U(y) is the sharp lower bound on the local error of an algorithm for every $y \in \mathfrak{N}(f)$. Since we want to assure that the local error of an algorithm φ is nearly as small as possible, we compare $e(\varphi,y)$ with rad U(y). This leads us to the concept of deviation defined as follows.

Definition 2.1

We shall say $\text{dev}(\varphi)$ is the deviation of the algorithm φ, $\varphi \in \Phi(\mathfrak{N},S,T)$, iff

$$(2.12) \quad \text{dev}(\varphi) = \sup_{y\in\mathfrak{N}(\mathfrak{J}_0)} \frac{e(\varphi,y)}{\text{rad}\, U(y)} \left(= \sup_{y\in\mathfrak{N}(\mathfrak{J}_0)} \sup_{f\in V(y)} \frac{\|\varphi(y)-Sf\|}{\text{rad}\, U(y)} \right). \quad \blacksquare$$

Of course, $\text{dev}(\varphi) \geq 1$ for any φ.

We mention two classes of algorithms for which the deviation is small. The first one is the class of central algorithms Φ^c defined by Traub and Woźniakowski [77]. An algorithm φ is central if $\varphi(y)$ is a center (if it exists) of the set U(y), $\forall y \in \mathfrak{N}(\mathfrak{J}_0)$. The center, $\varphi(y)$, is defined by

$$(2.13) \quad \sup_{x\in U(y)} \|x-\varphi(y)\| = \text{rad}\, U(y).$$

Obviously $\mathrm{dev}(\varphi) = 1$ iff φ is a central algorithm. A central algorithm is optimal but an optimal algorithm is not, in general, central.

The second class is the class of interpolatory algorithms Φ^I. By $\varphi \in \Phi^I$ we mean

$$(2.14) \qquad \varphi(y) = SG(y)$$

where $G: D_\varphi \to \mathfrak{J}_0$ and $\mathfrak{N}(G(y)) = y$. Note that G is a right inverse of $\mathfrak{N}$ and the element $G(y)$ belongs to $\mathfrak{J}_0$. Therefore, $G(y) \in V(y)$ and $\varphi(y) \in U(y)$. Then $e(\varphi,y) \le \mathrm{diam}\, U(y) \le 2\, \mathrm{rad}\, U(y)$ for any $\varphi \in \Phi^I$ where $\mathrm{diam}\, U(y) = \sup_{a,b \in U(y)} \|a-b\|$ is the diameter of $U(y)$. From this we conclude that $\mathrm{dev}(\varphi) \le 2$, $\forall \varphi \in \Phi^I$.

We summarize the properties of central and interpolatory algorithms.

Theorem 2.2

For any $y \in \mathfrak{N}(\mathfrak{J}_0)$

$$(2.15) \qquad \left.\begin{array}{l} e(\varphi,y) = \mathrm{rad}\, U(y), \\ \mathrm{dev}(\varphi) = 1, \end{array}\right\} \quad \forall \varphi \in \Phi^c,$$

and

$$(2.16) \qquad \left.\begin{array}{l} \mathrm{rad}\, U(y) \le e(\varphi,y) \le 2\, \mathrm{rad}\, U(y), \\ \mathrm{dev}(\varphi) \le 2, \end{array}\right\} \quad \forall \varphi \in \Phi^I. \qquad \blacksquare$$

As we mentioned in the Introduction we are also interested in algorithms with small combinatory complexity. There exists a class of algorithms for which the combinatory complexity is nearly minimal. This is the class of linear algorithms Φ^L. By $\varphi \in \Phi^L$ we mean a linear operator φ, i.e.,

$$(2.17)\quad \varphi(y) = \sum_{i=1}^{n} L_i(f) g_i, \qquad y = \mathfrak{N}(f) = [L_1(f), L_2(f), \ldots, L_n(f)],$$

where $g_i = g_i(\mathfrak{N}, S, T) \in \mathfrak{F}_2$. Since $g_1, g_2, \ldots, g_n$ are independent of f, they can be precomputed. Knowing g_i, we perform at most n multiplications of an element from $\mathfrak{F}_2$ by a scalar and (n-1) additions of elements from $\mathfrak{F}_2$ to compute $\varphi(y)$. Taking the cost of one such multiplication and one such addition as unity and neglecting the cost of precomputing $g_1, g_2, \ldots, g_n$ we conclude that the combinatory complexity of a linear algorithm is at most 2n-1.

The problem of linear algorithms with small deviation is closely related to spline algorithms which will be introduced in Section 4. Before that we remind the reader of the definition of splines in a linear normed space.

3. SPLINES

We remind the reader of the definition and some basic properties of splines in linear normed spaces and introduce notation we will use in this paper. See among others Anselone and Laurent [68], Atteia [65] and Holmes [72].

Let $y \in \mathbb{C}^n$. Define

$$(3.1) \quad A(y) = \{f \in \mathfrak{J}_1 : \mathfrak{N}(f) = y\}.$$

Note that for any y, the set A(y) is nonempty since $\mathfrak{N}$ is defined by n linearly independent linear functionals, i.e., $\mathfrak{N}(\mathfrak{J}_1) = \mathbb{C}^n$. See Section 2.

<u>Definition 3.1</u>

An element $\sigma(y) \in A(y)$ is called a <u>spline interpolating y</u> (briefly σ is a <u>spline</u>) iff

$$(3.2) \quad \|T\sigma(y)\| = \min_{f \in A(y)} \|Tf\| .$$

Let $z \in \mathfrak{J}_4$. Define

$$(3.3) \quad P(z) = \{h \in \ker \mathfrak{N} : \|Th-z\| = \operatorname{dist}(T(\ker \mathfrak{N}), z)\}.$$

Thus, every element of T(P(z)) is a best approximation of the element z from the set T(ker $\mathfrak{N}$). It is easy to observe that the concepts of splines and best approximation are closely connected. Namely, the following relations hold:

(3.4) There exists a spline $\sigma(y)$ interpolating y iff the set P(Tf) is nonempty for some $f \in A(y)$.

(3.5) An element $\sigma \in A(y)$ is a spline interpolating y iff $f-\sigma \in P(Tf)$ for every $f \in A(y)$.

(3.6) There exists a unique spline $\sigma(y)$ iff $\ker \mathfrak{N} \cap \ker T = \{0\}$ and $P(Tf)$ is a singleton set (i.e., $P(Tf)$ has exactly one element), $\forall f \in A(y)$.

Splines are homogenous, i.e., if $\sigma(y)$ is a spline interpolating y, then $c\sigma(y)$ is a spline interpolating cy for any constant $c \in \mathbb{C}$. This means that $\sigma(cy) = c\sigma(y)$ whenever the spline $\sigma(y)$ is uniquely defined.

Suppose that $\mathfrak{J}_4 = T(\mathfrak{J}_1)$ is a Hilbert space. Then $P(Tf)$ is nonempty for any $f \in \mathfrak{J}_1$ iff $T(\ker \mathfrak{N})$ is closed. Furthermore, $\sigma(y)$ is a spline iff $\sigma(y) \in A(y)$ and $(T\sigma(y), Th) = 0$, $\forall h \in \ker \mathfrak{N}$. A spline σ depends linearly on y, i.e., if splines $\sigma(y_1)$ and $\sigma(y_2)$ interpolate y_1 and y_2 respectively then $c_1\sigma(y_1) + c_2\sigma(y_2)$ is a spline interpolating $c_1y_1 + c_2y_2$ for any constants $c_1, c_2 \in \mathbb{C}$. The splines $\sigma(y)$ are uniquely defined iff $\ker \mathfrak{N} \cap \ker T = \{0\}$.

4. SPLINE ALGORITHMS

In this section we introduce the concept of spline algorithms and prove their optimality properties in the class of homogenous and interpolatory algorithms. To assure the existence of splines we shall assume throughout the rest of this paper that P(Tf) is a nonempty set for any $f \in \mathfrak{J}_1$.

Definition 4.1

We shall say φ is a spline algorithm, $\varphi \in \Phi^s$, iff

(4.1) $\varphi(y) = S\sigma(y), \quad \forall y \in \mathfrak{N}(\mathfrak{J}_0)$,

where $\sigma(y)$ is a spline interpolating y. ■

Note that a spline algorithm is interpolatory which implies that $e(\varphi,y) \le \text{diam}\, U(y) \le 2\, \text{rad}\, U(y)$, and $\text{dev}(\varphi) \le 2$. Since splines are homogenous, it is obvious that a spline algorithm is also homogenous.

Remark 4.1

To compute $\varphi(y)$ given y, in general we need to know a spline $\sigma(y)$, i.e., to solve the optimization problem (3.2). The complexity of solving (3.2) can be high. However if a spline algorithm is linear, i.e., $\varphi(\mathfrak{N}(f)) = \sum_{i=1}^{n} L_i(f)g_i$, then we need to compute only elements $g_1,g_2,\ldots,g_n$. Since the elements $g_1,g_2,\ldots,g_n$ are independent of $\mathfrak{N}(f)$, the idea of precomputing can be used. This means that in many cases we have to solve the optimization problem (3.2) only once. For nonlinear spline algorithms the idea of precomputing cannot in general be used and the combinatory complexity of such algorithms can be very high. ■

We establish the optimality properties of spline algorithms in the class $\Phi = \Phi(\mathfrak{N},S,T)$.

Lemma 4.1

Let $\varphi, \varphi \in \Phi$, be a homogenous algorithm and $y \in \mathfrak{N}(\mathfrak{F}_0)$. If $e(\varphi) < +\infty$ and $\sigma(y) \in \ker T$ then

$$(4.2) \quad \varphi(y) = S\sigma(y). \qquad \blacksquare$$

Proof

Since $\sigma(y) \in \ker T$ then $c\sigma(y) \in \mathfrak{F}_0$ for any $c \in \mathbb{C}$ and $\mathfrak{N}(c\sigma(y)) = cy$. Consider $\varphi(cy)-S(c\sigma(y)) = c(\varphi(y)-S\sigma(y))$. Observe that $|c| \, \|\varphi(y)-S\sigma(y)\| \le (\varphi) < +\infty$, $\forall c \in \mathbb{C}$. This implies $\varphi(y) = S\sigma(y)$ which proves (4.2). $\blacksquare$

emma 4.2

Let $\varphi, \varphi \in \Phi$, be a homogenous algorithm which is interpolatory. Then φ is a spline algorithm. $\blacksquare$

Proof

Since φ is interpolatory then $e(\varphi) \le 2r(\mathfrak{N},S,T) < +\infty$. Take an arbitrary y from $\mathfrak{N}(\mathfrak{F}_0)$ and consider a spline $\sigma(y)$. If $\sigma(y) \in \ker T$ then Lemma 4.1 yields $\varphi(y) = S\sigma(y)$. Thus we can assume that $T\sigma(y) \neq 0$. Let $\tilde{y} = y/\|T\sigma(y)\|$ and $\sigma(\tilde{y})$ be a spline which interpolates $\tilde{y}$. Note that $\|T\sigma(\tilde{y})\| = 1$ which implies that $\sigma(\tilde{y}) \in \mathfrak{F}_0$ and $\mathfrak{N}(\sigma(\tilde{y})) = \tilde{y}$. Consider the set $V(\tilde{y}) = \{f \in \mathfrak{F}_1 : \mathfrak{N}(f) = \tilde{y}, \|Tf\| \le 1\}$. Since $\sigma(\tilde{y}) \in V(y)$ and $1 = \|T\sigma(\tilde{y})\| \le \|Tf\|$ for any $f \in V(\tilde{y})$, every element of $V(\tilde{y})$ is a spline interpolating $\tilde{y}$.

The algorithm φ is interpolatory. Thus $\varphi(\tilde{y}) \in U(\tilde{y}) = SV(\tilde{y})$ which means that $\varphi(\tilde{y}) = S\sigma(\tilde{y})$ for a spline $\sigma(\tilde{y})$. Since φ is also homogenous, $\varphi(y) = \|T\sigma(y)\|\,\varphi(\tilde{y}) = \|T\sigma(y)\|\,S\sigma(\tilde{y}) = S\sigma(y)$ where $\sigma(y)$ is a spline which interpolates y. This proves that φ is a spline algorithm. ∎

We wish to examine when there exists a unique spline algorithm. It is easy to prove the following lemma.

Lemma 4.3

There exists a unique spline algorithm iff $SP(Tf)$ is a singleton set for any $f \in \mathfrak{J}_0$.

Proof

Let $f \in \mathfrak{J}_0$ and $y = \mathfrak{N}(f)$. Consider the splines $\sigma_1(y)$ and $\sigma_2(y)$. From Section 3 we know that $f-\sigma_i(y) = h_i \in P(Tf)$ for $i = 1,2$. Then

$$S\sigma_1(y)-S\sigma_2(y) = Sh_2-Sh_1.$$

Thus $SP(Tf)$ is singleton iff $S\sigma_1(y) = S\sigma_2(y)$, $\forall y \in \mathfrak{N}(\mathfrak{J}_0)$. This proves Lemma 4.3. ∎

We are ready to consider the deviation of homogenous algorithms belonging to Φ.

Theorem 4.1

Let $SP(Tf)$ be a singleton set for any $f \in \mathfrak{J}_0$. Let $\varphi, \varphi \in \Phi$, be a homogenous algorithm. Then

$$(4.3)\quad \mathrm{dev}(\varphi) = \begin{cases} \le 2 & \text{if } \varphi \text{ is a spline algorithm,} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof

The inequality $\mathrm{dev}(\varphi) \le 2$ for a spline algorithm φ was already proven. If $e(\varphi) = +\infty$ then $\mathrm{dev}(\varphi) = +\infty$ since $\mathrm{rad}\, U(y) \le r(\mathfrak{N},S,T) < +\infty$. Thus without loss of generality we can assume φ is a nonspline algorithm with $e(\varphi) < +\infty$. This means there exists $y \in \mathfrak{N}(\mathfrak{J}_0)$ such that $\varphi(y) \ne S\sigma(y)$ where $\sigma(y)$ is the unique spline interpolating y. Lemma 4.1 guarantees that $T\sigma(y) \ne 0$. As in the proof of Lemma 4.2 define $\tilde{y} = y/\|T\sigma(y)\|$ and consider the singleton set $U(\tilde{y}) = \{S\sigma(\tilde{y})\}$. Of course, $\mathrm{rad}\, U(\tilde{y}) = 0$. Since φ is homogenous,

$$\varphi(\tilde{y}) = \varphi(y)/\|T\sigma(y)\| \ne S\sigma(y)/\|T\sigma(y)\| = S\sigma(\tilde{y}).$$

Thus $e(\varphi,\tilde{y}) \ne 0$ and $\mathrm{dev}(\varphi) \ge e(\varphi,\tilde{y})/\mathrm{rad}\, U(\tilde{y}) = +\infty$. This completes the proof. $\square$

The assumption that $SP(Tf)$ is singleton for any $f \in \mathfrak{J}_0$ is essential. To see that consider the following example.

Example 4.1

Let $\mathfrak{J}_1 = \mathfrak{J}_2 = \mathfrak{J}_4 = \mathbb{C}[0,1]$ be the class of continuous functions on $[0,1]$ with the sup norm $\|f\| = \max_{0\le x\le 1} |f(x)|$. Let $S = T = I$ be the identity operator and $y = \mathfrak{N}(f) = [f(x_1), f(x_2), \ldots, f(x_n)]$ for some distinct points $x_i \in [0,1]$. Thus we want to recover the function f knowing its n values at x_i and the bound $\|f\| \le 1$. It is easy to show that

$$\mathrm{rad}\, U(y) = 1, \quad \forall y \in \mathfrak{N}(\mathfrak{J}_0) = [-1,1]^n,$$

and the unique center of $U(y)$ is the zero function. Note that the center of $U(y)$ does not belong to $U(y)$ for $y \ne 0$. Furthermore, every function σ, $\sigma \in \mathfrak{J}_1$, which agrees with f at x_i, i.e., $\sigma(x_i) = f(x_i) = y_i$, $i = 1,2,\ldots,n$,

and $\|\sigma\| \le \max_{1\le i\le n} |f(x_i)| = \|y\|_\infty$ is a spline. Thus if $f \notin \ker \mathfrak{N}$, i.e., $y \neq 0$, then there exist infinitely many splines and of course $SP(Tf) = P(f) = \{h \in \ker \mathfrak{N}: \|h-f\| \le \|y\|_\infty\}$ is not a singleton set.

Consider the central linear algorithm $\varphi(y) = 0$. Of course $dev(\varphi) = 1$. Since $\varphi(y) \equiv 0$ is not interpolatory, φ is not a spline algorithm. Furthermore it can be shown that any interpolatory algorithm φ has the local error $e(\varphi,y) = 2$ for any $y \in \mathfrak{N}(\mathfrak{J}_0)$, $\|y\|_\infty = 1$, and $dev(\varphi) = 2$. ■

<u>Remark 4.2</u>

Theorem 4.1 states that among homogenous algorithms only the spline algorithm has finite deviation. This provides the answer to question (1.1). When the spline algorithm is nonlinear the class of linear algorithms with finite deviation is empty. When the spline algorithm is linear, the class of linear algorithms with finite deviation consists of exactly one element; namely the unique linear spline algorithm. ■

Therefore it is important to know when a spline algorithm is linear. Although we defined a spline algorithm φ only for $y \in \mathfrak{N}(\mathfrak{J}_0)$, it is obvious that $\varphi(y) = S\sigma(y)$ where $\sigma(y)$ is a spline interpolating y for $y \in \mathbb{C}^n$, is the needed extension. Assume that $SP(Tf)$ is a singelton set for $f \in \mathfrak{J}^0$. Note that the set $P(z)$ defined by (3.3) is homogenous. Thus, since $SP(z)$ is a singleton set for any $z \in T(\mathfrak{J}_0)$, $SP(z) = \{a(z)\}$ is also singleton for any $z \in T(\mathfrak{J}_1) = \mathfrak{J}_4$. Therefore we can define an operator $R: \mathfrak{J}_4 \to \mathfrak{J}_2$ such that

(4.4) $R(z) = a(z)$.

The spline algorithm can be represented by R as

(4.5) $S\sigma(y) = Sf - R(Tf), \quad \forall y = \mathfrak{N}(f)$.

Indeed, since $\sigma(y)$ is a spline then $f - \sigma(y) \in P(Tf)$ for any f such that $\mathfrak{N}(f) = y$. Then $S(f-\sigma(y)) \in SP(Tf) = \{a(Tf)\}$ and $S\sigma(y) = Sf - S(f-\sigma(y)) = Sf - R(Tf)$ which proves (4.5). From (4.5) we immediately get

Lemma 4.4

Let SP(z) be a singleton set for any $z \in \mathfrak{J}_4$. Then the spline algorithm is linear iff R is a linear operator. ◻

Elsewhere in this paper we give examples for which a linear spline algorithm exists. We now illustrate Lemma 4.4 by an example of a unique spline algorithm which is nonlinear.

Example 4.2

Let $\mathfrak{J}_1 = \mathfrak{J}_2 = \mathfrak{J}_4$ be the space of polynomials of one variable of degree $\le n$. Define $\|f\| = \max_{0\le t\le 1} |f(t)|$ and let $S = T = I$. The information operator is given by

$$y = \mathfrak{N}(f) = [f'(0), \frac{f''(0)}{2!}, \ldots, \frac{f^{(n)}(0)}{n!}].$$

Thus $h \in \ker \mathfrak{N}$ implies $h(t) = \text{const}$. Then

$$P(f) = \{h(t) \equiv h_0 : \sup_{0\le t\le 1} |f(t)-h_0| = \inf_{c\in R} \sup_{0\le t\le 1} |f(t)-c|\},$$

i.e., $h \in P(f)$ iff h is a constant function and h is the best approximation of f among all zeroth degree polynomials. It is well known that $P(f) = \{a(f)\}$ is singleton and

$$R(f) = a(f) = (\bar{f}+\underline{f})/2$$

where $\bar{f} = \max_{0\le t\le 1} f(t)$ and $\underline{f} = \min_{0\le t\le 1} f(t)$. Of course, R is nonlinear which means

that the unique spline algorithm is also nonlinear. Define $g(y) = \sum_{i=1}^{n} y_i t^i$ where $y_i = \frac{f^{(i)}(0)}{i!}$. From (4.5) we get

$$\varphi^s(y) = S\sigma(y) = f - R(f) = g(y) + f(0) - (\bar{f}+\underline{f})/2 =$$
$$= g(y) + f(0) - (\bar{g}+2f(0)-\underline{g}) = g(y) - (\bar{g}+\underline{g})/2.$$

It can be shown that the spline algorithm is central and

$$e(\varphi^s,y) = 1 - (\bar{g}-\underline{g})/2 \le r(\mathfrak{N},S,T) = 1. \quad \blacksquare$$

5. SPLINE ALGORITHMS IN A HILBERT CASE

In this section we assume that the operator T maps onto a Hilbert space $\mathfrak{F}_4 = T(\mathfrak{F}_1)$ and $T(\ker \mathfrak{N})$ is closed. Observe that the set $SP(Tf)$ is singleton for any $f \in \mathfrak{F}_1$. Indeed let $\sigma_1(y)$ and $\sigma_2(y)$ be splines which interpolate $y = \mathfrak{N}(f)$. Then $T\sigma_i(y)$ is orthogonal to $T(\ker \mathfrak{N})$. Let $h = \sigma_1(y) - \sigma_2(y)$. Then $h \in \ker \mathfrak{N}$ and $\|T\sigma_1\|^2 = \|T\sigma_2\|^2 = \|T\sigma_1 - Th\|^2 = \|T\sigma_1\|^2 + \|Th\|^2$. This implies that $Th = 0$. Thus $h \in \ker \mathfrak{N} \cap \ker T$ and (2.11) yields $h \in \ker S$, i.e., $S\sigma_1(y) = S\sigma_2(y)$.

The unique spline algorithm may be derived as follows. Let $e^i = [0,\ldots,1,\ldots,0] \in \mathbb{C}^n$ denote the ith unit vector, $i=1,2,\ldots,n$. Find $\sigma_1 \in \mathfrak{F}_1$ such that $\mathfrak{N}(\sigma_i) = e^i$ and $T\sigma_i$ is orthogonal to $T(\ker \mathfrak{N})$. Of course σ_i is a spline interpolating e^i. Then $\sigma(y) = \sum_{i=1}^{n} L_i(f)\sigma_i$ is also orthogonal to $T(\ker \mathfrak{N})$ and therefore $\sigma(y)$ is a spline interpolating $y = [L_1(f), L_2(f), \ldots, L_n(f)]$. The unique spline algorithm φ^s is of the form

$$(5.1) \quad \varphi^s(y) = S\sigma(y) = \sum_{i=1}^{n} L_i(f) S\sigma_i$$

for $y = \mathfrak{N}(f)$. This shows that the spline algorithm is linear.

<u>Theorem 5.1</u>

If $\mathfrak{F}_4$ is a Hilbert space and $T(\ker \mathfrak{N})$ is closed then the spline algorithm φ^s is central and

$$(5.2) \quad e(\varphi^s, y) = \text{rad}\, U(y) = \sqrt{1 - \|T\sigma(y)\|^2}\quad r(\mathfrak{N}, S, T)$$

where the radius of information is equal to

$$(5.3)\quad r(\mathfrak{N},S,T) = \sup_{h\in \ker \mathfrak{N}} \frac{\|Sh\|}{\|Th\|} .$$

Proof

Let $f \in V(y)$. Then $f = \sigma(y) + h$ where $h \in \ker \mathfrak{N}$ and $1 \ge \|Tf\|^2 = \|T\sigma(y)\|^2 + \|Th\|^2$. Thus

$$V(y) = \{\sigma(y) + h : h \in \ker \mathfrak{N} \text{ and } \|Th\|^2 \le 1 - \|T\sigma(y)\|^2\}.$$

We now show that $U(y) = SV(y)$ is symmetric with respect to $S\sigma(y)$. Indeed, let $S\sigma(y) + Sh \in U(y)$. Then $h \in \ker \mathfrak{N}$ and $\|Th\|^2 \le 1 - \|T\sigma(y)\|^2$. The element $S\sigma(y) - Sh$ also belongs to $U(y)$ since $\mathfrak{N}(\sigma(y)-h) = y$ and $\|T(\sigma(y)-h)\|^2 = \|T\sigma(y)\|^2 + \|Th\|^2 \le 1$.

From this it easily follows that $a = S\sigma(y)$ is a center of $U(y)$. To show this assume that a is not a center, i.e., there exist elements $b \in U(y)$ and $s \in \mathfrak{J}_2$ such that

$$(5.4)\quad \sup_{u\in U(y)} \|s-u\| < \|b-a\| .$$

Since $U(y)$ is symmetric with respect to a and $a \in U(y)$ then $2a-b$ also belongs to $U(y)$ and (5.4) yields $\|s-(2a-b)\| < \|b-a\|$. But

$$2\|b-a\| = \|s-(2a-b)+b-s\| \le \|s-(2a-b)\| + \|s-b\| < 2\|b-a\|$$

which is a contradiction.

Hence, $\varphi^s(y) = S\sigma(y)$ is a central algorithm and

$$\begin{aligned} e(\varphi^s,y) &= \text{rad } U(y) = \sup\{\|S(\sigma(y)+h)-S\sigma(y)\| : \sigma(y) + h \in V(y)\} = \\ &= \sup\{\|Sh\| : h \in \ker \mathfrak{N},\ \|Th\| \le \sqrt{1-\|T\sigma(y)\|^2}\ \} = \\ &= \sqrt{1-\|T\sigma(y)\|^2}\ \sup\{\|Sh\|/\|Th\| : h \in \ker \mathfrak{N}\}. \end{aligned}$$

Observe that $r(\mathfrak{N},S,T) = \sup_{y\in\mathfrak{N}(\mathfrak{F}_0)} \text{rad } U(y) = \text{rad } U(0)$ which completes the proof. ∎

Theorem 5.1 states that the spline algorithm is central. It is also linear. These are very desirable and useful properties. This affirmatively answers our questions from Section 1 for any linear operators S, T and $\mathfrak{N}$ (assuming that $\mathfrak{F}_4$ is a Hilbert space and $T(\ker \mathfrak{N})$ is closed). We illustrate Theorem 5.1 by two examples.

Example 5.1

Suppose that $\mathfrak{F}_1 = \mathfrak{F}_4$ is a Hilbert space with an orthonormal basis $\{\xi_i\}_{i=1}^{\infty}$. Let T be an orthogonal operator, i.e., $T^*T = TT^* = I$. Let $f \in \mathfrak{F}_1$, i.e., $f = \sum_{i=1}^{\infty} (f,\xi_i)\xi_i$. Define the information operator $\mathfrak{N}$ as

$$y = \mathfrak{N}(f) = [(f,\xi_1),(f,\xi_2),\ldots,(f,\xi_n)].$$

Then $\sigma(y) = \sum_{i=1}^{n} (f,\xi_i)\xi_i$ satisfies $\mathfrak{N}(\sigma(y)) = y$ and for any $h \in \ker \mathfrak{N} = \{h : (h,\xi_i) = 0,\ i=1,2,\ldots,n\}$ we get $(T\sigma(y),Th) = (\sigma,h) = \sum_{i=1}^{n} (f,\xi_i)(\xi_i,h) = 0.$ This shows that a truncated Fourier series $\sigma(y)$ is a spline. Thus for any linear operator $S : \mathfrak{F}_1 \to \mathfrak{F}_2$ the spline algorithm

$$\varphi(y) = \sum_{i=1}^{n} (f,\xi_i)S\xi_i$$

is central and $r(\mathfrak{N},S,T) = \|S\|_{\ker \mathfrak{N}} = \sup\{\|Sh\| : h \in \ker \mathfrak{N} \text{ and } \|h\| \le 1\}$. ∎

Example 5.2

Let $\mathfrak{F}_1 = W^{k,2}[0,1]$ be a Sobolev space, i.e., the space of functions for which the (k-1)-st derivative is absolutely continuous and the k-th derivative belongs to $\mathfrak{F}_4 = L^2[0,1]$. Let $T = D^k$, i.e., $Tf = f^{(k)}$, and

$$\mathfrak{N}(f) = [f(x_1),\ldots,f^{(j_1-1)}(x_1),\ldots,f(x_r),\ldots,f^{(j_r-1)}(x_r)]$$

for distinct $x_i \in [0,1]$ and $\max_{1\le i\le r} j_i < k$. Then the cardinality of $\mathfrak{N}$ is equal to $n = j_1+j_2+\ldots+j_r$. Assuming that $n \ge k$, it is well-known that the spline $\sigma(y)$ is the natural spline function of degree 2n-1 with respect to the knots $x_1,x_2,\ldots,x_r$ with multiplicity $j_1,j_2,\ldots,j_r$ respectively.

Many authors dealt with this information operator with different $j_1,j_2,\ldots,j_r$ for different linear operators S. See the Introduction for the reference list. For instance if $S = I$ or $S(f) = \int_0^1 f(x)dx$ it is known that $r(\mathfrak{N},S,T) = +\infty$ for $n < k$ and $r(\mathfrak{N},S,T) = 0(n^{-k})$ for $n \ge k$. Theorem 5.1 states that for any linear operator S the algorithm obtained through natural spline is central. ∎

6. SPLINE ALGORITHMS IN A NON-HILBERT CASE

In this section we deal with spline algorithms where $\mathfrak{J}_4$ is not necessarily a Hilbert space. We give necessary and sufficient conditions on a spline algorithm to be a central or optimal algorithm. We also show examples for which spline algorithms are neither central nor optimal. We begin with the centrality of spline algorithms. It is intuitively obvious that a spline algorithm is central iff the centers of $U(y)$ belong to $U(y)$ and enjoy the same homogenous property. A formal proof is provided by

<u>Lemma 6.1</u>

There exists a central spline algorithm φ^c iff there exists a function $c : \mathfrak{N}(\mathfrak{J}_0) \to \mathfrak{J}_2$ such that

(i) $c(y) \in U(y)$ and $c(y)$ is a center of $U(y)$, $\forall y \in \mathfrak{N}(\mathfrak{J}_0)$,

(ii) c is homogenous, i.e., $tc(y) \in U(ty)$ and is a center of $U(ty)$, $\forall |t| \le 1, \forall y \in \mathfrak{N}(\mathfrak{J}_0)$.

<u>Proof</u>

Suppose that φ^c is a central spline algorithm. Define $c(y) = \varphi^c(y) = S\sigma(y)$. Since φ^c is interpolatory, central and $\sigma(y)$ is homogenous then c has the desired properites.

Assume now that c satisfies (i) and (ii) which means $c(y)$ is a homogenous algorithm which is interpolatory. From Lemma 4.2 it follows that $c(y)$ is a spline algorithm. ∎

We now show an example where the unique linear spline algorithm is optimal but not central.

Example 6.1

Let $\mathfrak{F}_1 = \mathfrak{F}_2 = W^{1,+\infty}[0,1]$ be the space of functions which are absolutely continuous and $f' \in L_\infty[0,1]$. Consider $S = I$, $Tf = f'$ and $\mathfrak{N}(f) = [f(0), f(1)]$.

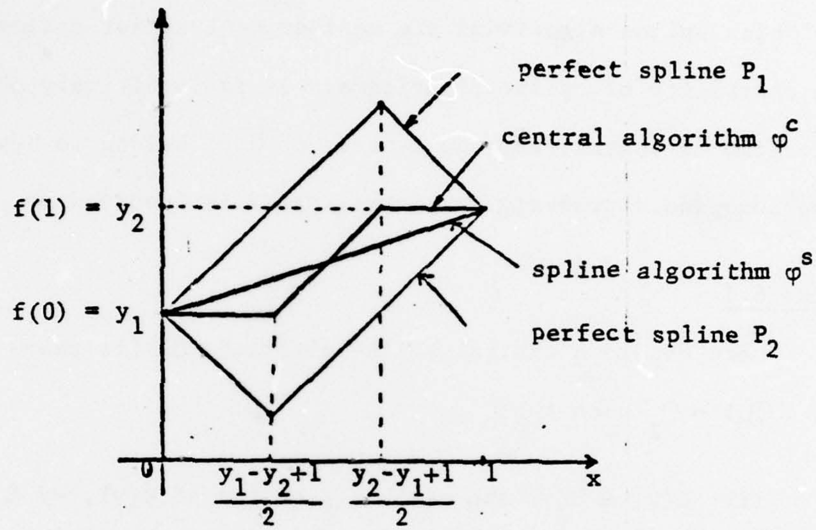

Fig. 6.1

It is easy to verify that there exists a unique spline algorithm

$$\varphi^s(y)(x) = y_1 + (y_2-y_1)x,$$

$$e(\varphi^s) = \frac{1}{2} = r(\mathfrak{N}, S, T).$$

But the central algorithm is equal to

$$\varphi^c(y)(x) = (p_1(x)+p_2(x))/2$$

where p_1 and p_2 are perfect splines,

$$p_1(x) = \begin{cases} x + y_1 & \text{for } x \le \frac{y_2-y_1+1}{2}, \\ -x + y_2 + 1 & \text{otherwise,} \end{cases}$$

$$p_2(x) = \begin{cases} -x + y_1 & \text{for } x \le \frac{y_1-y_2+1}{2} \\ x + y_2 - 1 & \text{otherwise} \end{cases}$$

and $\varphi^s \neq \varphi^c$. ◻

We turn to the question when a spline algorithm is optimal. Note that we now deal with nonunique spline algorithms since we do not assume that $SP(Tf)$ is a singleton set.

Lemma 6.2

There exists an optimal spline algorithm iff there exists a function $\sigma : \mathfrak{N}(\mathfrak{I}_0) \to \mathfrak{I}_2$ such that

(i) $\sigma(y)$ is a spline interpolating y, $\forall y$,

(ii) for any $f \in \mathfrak{I}_0$, $y = \mathfrak{N}(f)$,

$$\| Tf - T\sigma(y) \| > 1 \text{ implies } \| Sf - S\sigma(y) \| \le r(\mathfrak{N},S,T).$$ ■

Proof

Suppose that $\bar{\varphi}^s$ is an optimal spline algorithm, $\bar{\varphi}^s(y) = S\bar{\sigma}(y)$, where $\bar{\sigma}(y)$ is a spline interpolating y. Then $\| Sf - S\bar{\sigma}(y) \| \le r(\mathfrak{N},S,T)$ for any $f \in \mathfrak{I}_0$, $y = \mathfrak{N}(f)$. Hence we can put $\sigma(y) = \bar{\sigma}(y)$ which satisfies (i) and (ii).

Assume now that σ satisfies the conditions (i) and (ii). Define an algorithm $\varphi(y) = S\sigma(y)$. Due to (i), φ is a spline algorithm. Consider $\| Sf - \varphi(y) \|$ for $y = \mathfrak{N}(f)$. If $\| Tf - T\sigma(y) \| > 1$ then (ii) implies $\| Sf - \varphi(y) \| \le r(\mathfrak{N},S,T)$. If $\| Tf - T\sigma(y) \| \le 1$ then setting $h = f - \sigma(y)$, $h \in \ker \mathfrak{N}$, we have $\| Sf - \varphi(y) \| = \| Sh \|$ and $\| Th \| \le 1$. Thus $\| Sf - \varphi(y) \| \le$ $\le \| Sh \| / \| Th \| \le r(\mathfrak{N},S,T)$, due to (2.10). This means that φ is an optimal spline algorithm which completes the proof. ■

Lemma 6.2 states that a spline algorithm $\varphi^s(y) = S\sigma(y)$ is optimal if the elements $h = f - \sigma(y)$ of the large T-norm, $\| Th \| > 1$, are correlated with the operator S in such a way that $\| Sh \|$ is small, i.e., $\| Sh \| \le r(\mathfrak{N},S,T)$.

Note that if $\mathfrak{F}_4$ is a Hilbert space we have a unique spline algorithm for which $\|Th\| = \sqrt{\|Tf\|^2 - \|T\sigma(y)\|^2} \le \|Tf\| \le 1$ and condition (ii) is automatically satisfied. Note that Lemma 6.2 provides the answer to question (1.2). Namely, the class of linear algorithms with finite deviation which are optimal is nonempty iff there exists a function σ satisfying (i) and (ii) of Lemma 6.1.

Example 6.1 provides a problem in a non-Hilbert space $\mathfrak{F}_4$ for which condition (ii) holds. We now consider an example for which an unique spline algorithm is not optimal and its deviation is arbitrary close to two.

Example 6.2

Let $\mathfrak{F}_1 = \mathfrak{F}_2 = R^2 = \{(f_1,f_2) : f_i \text{ are real}\}$ with the l_2-norm $\|f\| = \sqrt{f_1^2+f_2^2}$. Let $\mathfrak{F}_4 = R^2$ be equipped with the norm $\|f\| = \max(|f_1|,\frac{1}{a}|f_2|)$ where a parameter $a \in (1,+\infty)$. Define $Sf = f$, $Tf = f$ and $\mathfrak{N}(f) = f_1+f_2$. Thus knowing the sum of components of f we want to recover f where $f \in \mathfrak{F}_0 = \{f : \|Tf\| = \max(|f_1|,\frac{1}{a}|f_2|) \le 1\}$.

Figure 6.2 shows the central and spline algorithms. The central algorithm φ^c is equal to

$$\varphi^c(y) = \begin{cases} \left(\frac{y+1-a}{2}, \frac{y-1+a}{2}\right) & \text{for } a-1 \le y \le a+1, \\ (0,y) & \text{for } |y| \le a-1, \\ \left(\frac{y-1+a}{2}, \frac{y+1-a}{2}\right) & \text{for } 1-a \le y \le -a+1, \end{cases}$$

and

$$e(\varphi^c,y) = \begin{cases} \sqrt{2} & \text{for } |y| \le a-1 \\ \frac{\sqrt{2}}{2}(1+a-|y|) & \text{otherwise.} \end{cases}$$

Hence $r(\mathfrak{N},S,T) = \sqrt{2}$. The unique spline algorithm φ^s is given by

$$\varphi^s(y) = y\left(\frac{1}{1+a}, \frac{a}{1+a}\right).$$

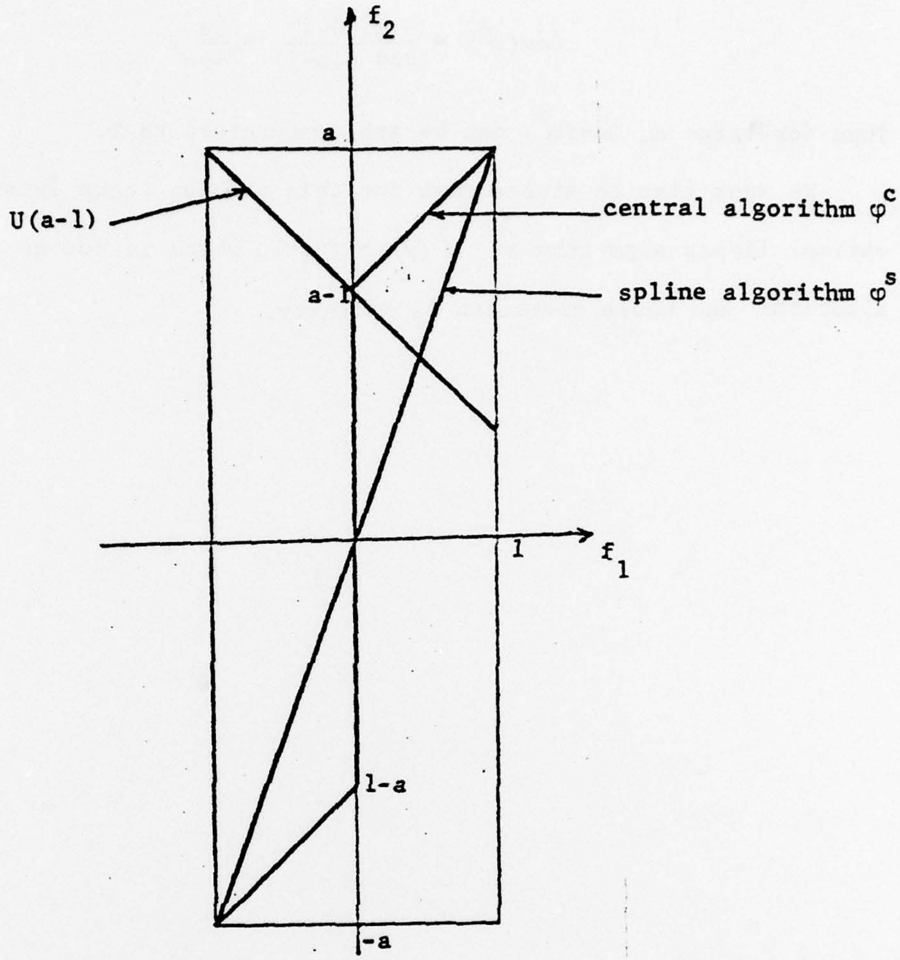

Fig. 6.2

The local error of φ^s is

$$e(\varphi^s,y) = \begin{cases} \frac{\sqrt{2}}{1+a}\,(y+a+1) & \text{for } |y| \le a-1, \\ \frac{\sqrt{2}}{1+a}\,a(1+a-|y|) & \text{otherwise,} \end{cases}$$

and $e(\varphi^s) = e(\varphi^s,a-1) = \frac{2\sqrt{2}}{1+a}\,a = \frac{2a}{1+a}\,r(\mathfrak{N},S,T)$. This means that φ^s is <u>not</u> optimal. Furthermore

$$\mathrm{dev}(\varphi^S) = \frac{e(\varphi^S, a-1)}{\mathrm{rad}\ U(a-1)} = \frac{2a}{1+a}.$$

Thus for large a, $\mathrm{dev}(\varphi^S)$ can be arbitrary close to 2.

We want also to stress that for this problem there exists a unique optimal linear algorithm φ^L, $\varphi^L(y) = (0,y)$, which is not an interpolatory algorithm and whose deviation is infinity. ∎

7. RELATION TO THE KOMOGOROV n-WIDTH

We briefly note that there is a relationship between the concept of the radius of information and the Kolmogorov n-width. For the definition of Kolmogorov n-width see, for instance, Lorentz [66]. Namely, suppose there exists a linear optimal error algorithm φ. Thus Sf, for any $f \in \mathfrak{I}_0$, is approximated by an element $\varphi(y) = \sum_{i=1}^{n} L_i(f) g_i \in G = \mathrm{lin}(g_1, g_2, \ldots, g_n)$. Note that $\dim G \le n$ and

$$(7.1)\quad r(\mathfrak{N},S,T) = e(\varphi) \ge \sup_{f \in \mathfrak{I}_0} \inf_{g \in G} \| Sf-g \| \ge$$

$$\ge \inf_{\substack{G \\ G \subset \mathfrak{I}_2, \dim G \le n}} \sup_{f \in \mathfrak{I}_0} \inf_{g \in G} \| Sf-g \| = d_n(S(\mathfrak{I}_0), \mathfrak{I}_2)$$

where $d_n(S(\mathfrak{I}_0), \mathfrak{I}_2)$ is the Kolmogorov n-width of the set $S(\mathfrak{I}_0)$ in the space $\mathfrak{I}_2$.

Furthermore if $\mathfrak{I}_4$ is a Hilbert space then it can be shown that

$$\inf_{\substack{\mathfrak{N} \\ \mathrm{card}(\mathfrak{N}) \le n}} r(\mathfrak{N},S,T) = d_n(S(\mathfrak{I}_0), \mathfrak{I}_2)$$

which means that for a suitably chosen information operator the radius is equal to the Kolmogorov n-widths. These and other relations with the Gelfand and linear Kolmogorov n-widths and entropy for linear problems in general linear spaces will be reported in a research monograph (Traub and Woźniakowski [79]).

8. SUMMARY

We summarize the results of the paper. Assuming the uniqueness of the spline algorithm φ^s, the answer to question (1.1) is that the class of linear algorithms with finite deviation is empty if φ^s is nonlinear. This class consists of one element, namely φ^s, if φ^s is linear. The answer to question (1.2) is that the class of linear algorithms with finite deviation which are optimal is empty if φ^s is nonlinear or nonoptimal. This class consists of one element, namely φ^s, if φ^s is linear and optimal. We give necessary and sufficiently conditions for φ^s to be linear, central, and optimal.

If $T(\ker \mathfrak{N})$ is closed in a Hilbert space, there exists a unique spline algorithm φ^s which is central and linear. Due to centrality it yields the best possible approximation for every f. Due to linearity, its combinatory complexity is small since the idea of precomputing can be used. We stress again that in general the combinatory complexity of a nonlinear spline algorithm is high and that precomputing cannot be used.

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